

RATIONAL MAP $ax + 1/x$ ON THE PROJECTIVE LINE OVER \mathbb{Q}_p

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ABSTRACT. The dynamical structure of the rational map $ax + 1/x$ on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ over the field \mathbb{Q}_p of p -adic numbers is described for $p \geq 3$.

1. INTRODUCTION

For a prime number p , let \mathbb{Q}_p be the field of p -adic numbers and $\mathbb{P}^1(\mathbb{Q}_p)$ its projective line. Recently, polynomials and rational maps of \mathbb{Q}_p have been studied as dynamical systems on \mathbb{Q}_p or $\mathbb{P}^1(\mathbb{Q}_p)$. It turns out that these p -adic dynamical systems are quite different to the dynamical systems in Euclidean spaces. See for example, [2, 5, 26] and their bibliographies therein.

For polynomials and rational maps of \mathbb{Q}_p , we can find two different kinds of subsystems exhibiting totally different dynamical behavior. One is 1-Lipschitz dynamical systems and the other is p -adic repellers.

A 1-Lipschitz p -adic dynamical system can usually be fully described by showing all its minimal subsystems. Polynomials with coefficients in the ring \mathbb{Z}_p of p -adic integers and rational maps with good reduction are two important families of 1-Lipschitz dynamical systems. In [14], the authors proved the following structure theorem for polynomials in $\mathbb{Z}_p[x]$. The same structure theorem for good reduction maps with degree at least 2 was proved in [12].

Theorem 1 ([14], Theorem 1). *Let $f \in \mathbb{Z}_p[x]$ be a polynomial of integral coefficients with degree ≥ 2 . We have the following decomposition*

$$\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B}$$

where \mathcal{P} is the finite set consisting of all periodic points of f , $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each \mathcal{M}_i is a finite union of balls and each subsystem $f : \mathcal{M}_i \rightarrow \mathcal{M}_i$ is minimal, and each point in \mathcal{B} lies in the attracting basin of a periodic orbit or of a minimal subsystem.

The decomposition in Theorem 1 is usually referred to as a *minimal decomposition* and the invariant subsets \mathcal{M}_i are called *minimal components*. In the literature, the minimality of the polynomial (or 1-Lipschitz) dynamical systems on the whole space \mathbb{Z}_p was widely studied [1, 3, 7, 10, 13, 17, 18, 20, 24].

The p -adic repellers (see definition in page 220 of [15]) are expanding dynamical systems which have positive topological entropy and thus exhibit chaotic behaviors. In [15],

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it is proved that a transitive p -adic repeller is isometrically (hence topologically) conjugate to a subshift of finite type where a suitable metric is defined. A general method was also proposed in [15] to find subshifts of finite type subsystems in a p -adic polynomial dynamical system. We remark that Thiran, Verstegen and Weyers [28] and Dremov, Shabat and Vytynova [9] studied the chaotic behavior of p -adic quadratic polynomial dynamical systems. Woodcock and Smart [29] proved that the so-called p -adic logistic map $\frac{x^p - x}{p}$ is topologically conjugate to the full shift on the symbolic system with p symbols.

On the other side, the dynamical properties of the fixed points of the rational maps have been studied in the space \mathbb{C}_p of p -adic complex numbers [4, 21, 23, 27] and in the adelic space [8]. The Fatou and Julia theory of the rational maps on \mathbb{C}_p , and on the Berkovich space over \mathbb{C}_p , are also developed [6, 5, 19, 25, 26]. However, the global dynamical structure of rational maps on \mathbb{Q}_p remains unclear, though the rational maps of degree one are totally characterized in [11].

In the present article, we suppose $p \geq 3$ and investigate the following special class of rational maps of degree 2:

$$\phi(x) = ax + \frac{1}{x}, \quad a \in \mathbb{Q}_p \setminus \{0\}. \quad (1.1)$$

We distinguish three cases: (1) $|a|_p = 1$, (2) $|a|_p > 1$, (3) $|a|_p < 1$.

Observe that

$$\phi(x) = ax + \frac{1}{x} = \frac{ax^2 - 1}{x}.$$

When $|a|_p = 1$, the map ϕ has good reduction (see definition in page 58 of [26]). By Theorem 1.2 of [12], we immediately have the following structure theorem.

Theorem 2. *Let $\phi(x) = ax + \frac{1}{x}$ with $a \in \mathbb{Q}_p \setminus \{0\}$. If $|a|_p = 1$, the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ can be decomposed into*

$$\mathbb{P}^1(\mathbb{Q}_p) = \mathcal{P} \bigsqcup \mathcal{M} \bigsqcup \mathcal{B}$$

where \mathcal{P} is the finite set consisting of all periodic points of ϕ , $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each \mathcal{M}_i is a finite union of balls and each subsystem $\phi : \mathcal{M}_i \rightarrow \mathcal{M}_i$ is minimal, and each point in \mathcal{B} lies in the attracting basin of a periodic orbit or of a minimal subsystem.

We are thus left to study the rest two cases. The following are our main theorems. We remark that in both cases, $\infty \in \mathbb{P}^1(\mathbb{Q}_p)$ is a fixed point of ϕ with multiplier $1/a$.

Theorem 3. *Let $\phi(x) = ax + \frac{1}{x}$ with $a \in \mathbb{Q}_p \setminus \{0\}$. If $|a|_p > 1$, the dynamical structure of the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is described as follows.*

(1) *If $\sqrt{1-a} \notin \mathbb{Q}_p$, then*

$$\forall x \in \mathbb{Q}_p, \quad \lim_{n \rightarrow \infty} \phi^n(x) = \infty.$$

(2) *If $\sqrt{1-a} \in \mathbb{Q}_p$, then there exists an invariant set \mathcal{J} such that the subsystem (\mathcal{J}, ϕ) is topologically conjugate to (Σ_2, σ) , the full shift of two symbols. Further,*

$$\forall x \in \mathbb{Q}_p \setminus \mathcal{J}, \quad \lim_{n \rightarrow \infty} \phi^n(x) = \infty.$$

Theorem 4. *Let $\phi(x) = ax + \frac{1}{x}$ with $a \in \mathbb{Q}_p \setminus \{0\}$. If $|a|_p < 1$, we distinguish two cases.*

(1) *If $\sqrt{-a} \notin \mathbb{Q}_p$, then $\phi(0) = \phi(\infty) = \infty$ is a repelling fixed point and the subsystem $(\mathbb{Q}_p \setminus \{0\}, \phi)$ has a minimal decomposition as stated in Theorem 2.*

- (2) If $\sqrt{-a} \in \mathbb{Q}_p$, then the dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ has a subsystem which is conjugate to a subshift of finite type with positive entropy.

2. PRELIMINARIES

Let $p \geq 2$ be a prime number. Any nonzero rational number $r \in \mathbb{Q}$ can be written as $r = p^v \frac{a}{b}$ where $v, a, b \in \mathbb{Z}$ and a, b are not divisible by p . Define $v_p(r) = v$ and $|r|_p = p^{-v_p(r)}$ for $r \neq 0$ and $|0|_p = 0$. Then $|\cdot|_p$ is a non-Archimedean absolute value on \mathbb{Q} . That means

- (i) $|r|_p \geq 0$ with equality only for $r = 0$;
- (ii) $|rs|_p = |r|_p |s|_p$;
- (iii) $|r + s|_p \leq \max\{|r|_p, |s|_p\}$.

The field \mathbb{Q}_p of p -adic numbers is the completion of \mathbb{Q} under the absolute value $|\cdot|_p$. Actually, any $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{n=v_p(x)}^{\infty} a_n p^n \quad (v_p(x) \in \mathbb{Z}, a_n \in \{0, 1, 2, \dots, p-1\} \text{ and } a_{v_p(x)} \neq 0).$$

Here, the integer $v_p(x)$ is called the p -valuation of x .

Any point in the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ of \mathbb{Q}_p can be given in homogeneous coordinates by a pair $[x_1 : x_2]$ of points in \mathbb{Q}_p which are not both zero. Two such pairs are equal if they differ by an overall (nonzero) factor $\lambda \in \mathbb{Q}_p^*$:

$$[x_1 : x_2] = [\lambda x_1 : \lambda x_2].$$

The field \mathbb{Q}_p is identified with the subset of $\mathbb{P}^1(\mathbb{Q}_p)$ given by

$$\{[x : 1] \in \mathbb{P}^1(\mathbb{Q}_p) \mid x \in \mathbb{Q}_p\}.$$

This subset contains all points in $\mathbb{P}^1(\mathbb{Q}_p)$ except one: the point of infinity, which may be given as $\infty = [1 : 0]$.

The spherical metric defined on $\mathbb{P}^1(\mathbb{Q}_p)$ is analogous to the standard spherical metric on the Riemann sphere. If $P = [x_1, y_1]$ and $Q = [x_2, y_2]$ are two points in $\mathbb{P}^1(\mathbb{Q}_p)$, we define

$$\rho(P, Q) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}$$

or, viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $\mathbb{Q}_p \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \quad \text{if } z_1, z_2 \in \mathbb{Q}_p,$$

and

$$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \leq 1, \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$$

Remark that the restriction of the spherical metric on the ring $\mathbb{Z}_p := \{x \in \mathbb{Q}_p, |x|_p \leq 1\}$ of p -adic integers is the same as the metric induced by the absolute value $|\cdot|_p$.

A rational map $\phi \in \mathbb{Q}_p(z)$ induces a transformation on $\mathbb{P}^1(\mathbb{Q}_p)$. Rational maps are always Lipschitz continuous on $\mathbb{P}^1(\mathbb{Q}_p)$ with respect to the spherical metric (see [26, Theorem 2.14.]).

In \mathbb{Q}_p , we denote by $D(a, r) := \{x \in \mathbb{Q}_p : |x|_p \leq r\}$ the closed disk centered at a with radius r and by $S(a, r) := \{x \in \mathbb{Q}_p : |x|_p = r\}$ its corresponding sphere. A closed disk in $\mathbb{P}^1(\mathbb{Q}_p)$ is either a closed disk in \mathbb{Q}_p or the complement of an open disk in \mathbb{Q}_p .

We recall some standard terminology of the theory of dynamical systems. If $\phi(x_0) = x_0$ then x_0 is called a fixed point of ϕ . The set of all fixed points of f is denoted by $\text{Fix}(f)$. An important role in iteration theory is played by the periodic points. By definition, x_0 is called a *periodic point* of ϕ if $\phi^n(x_0) = x_0$ for some $n \geq 1$. In this case, n is called a *period* of x_0 , and the smallest n with this property is called the *exact period* of x_0 .

For a periodic point $x_0 \in \mathbb{Q}_p$ of exact period n , $(\phi^n)'(x_0)$ is called the multiplier of x_0 . Remark that the multiplier is invariant by changing of coordinate. If ∞ is a periodic point of period n , then the multiplier of ∞ is $\psi'(0)$, where $\psi(x) = \frac{1}{\phi^n(1/x)}$. A periodic point is called *attracting*, *indifferent*, or *repelling* accordingly as the absolute value of its multiplier is less than, equal to, or greater than 1. Periodic points of multiplier 0 are called *super attracting*.

A subsystem of a dynamical system is *minimal* if the orbit of any point in the subspace is dense in the subspace.

Now we recall the conditions under which a number in \mathbb{Q}_p has a square root in \mathbb{Q}_p .

An integer $a \in \mathbb{Z}$ is called a *quadratic residue modulo p* if the equation $x^2 \equiv a \pmod{p}$ has a solution $x \in \mathbb{Z}$. The following lemma characterizes those p -adic integers which admit a square root in \mathbb{Q}_p .

Lemma 1 ([22]). *Let a be a nonzero p -adic number with its p -adic expansion*

$$a = p^{v_p(a)}(a_0 + a_1p + a_2p^2 + \cdots)$$

where $1 \leq a_0 \leq p-1$ and $0 \leq a_j \leq p-1$ ($j \geq 1$). The equation $x^2 = a$ has a solution $x \in \mathbb{Q}_p$ if and only if the following conditions are satisfied

- (i) $v_p(a)$ is even;
- (ii) a_0 is quadratic residue modulo p if $p \neq 2$; or $a_1 = a_2 = 0$ if $p = 2$.

3. DYNAMICAL STRUCTURES

Now, let $p \geq 3$. We will study the dynamical structure of the rational maps $\phi(x) = ax + \frac{1}{x}$ with $a \in \mathbb{Q}_p \setminus \{0\}$ on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$. In general, we can also consider the rational maps

$$ax + \frac{b}{x}, \quad \text{with } a, b \in \mathbb{Q}_p.$$

Remark that if \sqrt{b} exists in \mathbb{Q}_p , then $ax + \frac{b}{x}$ is conjugate to $ax + \frac{1}{x}$ through the conjugacy $x \mapsto \frac{1}{\sqrt{b}}x$. The case that \sqrt{b} does not exist in \mathbb{Q}_p which is not included in the present paper could be a subject for future study.

The dynamical system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ exhibits different dynamical structures according to different absolute values of a . When $|a|_p = 1$, the transformation ϕ has good reduction. The dynamical structure of $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$, as shown in Theorem 2, can be deduced directly from Theorem 1.2 of [12].

We are thus concerned only with the cases: $|a|_p > 1$ and $|a|_p < 1$. We remark that for both cases, ∞ is a fixed point of ϕ .

3.1. Case $|a|_p > 1$.

Proposition 1. *Suppose $|a|_p > 1$. If $\sqrt{1-a} \notin \mathbb{Q}_p$, then*

$$\forall x \in \mathbb{Q}_p, \quad \lim_{n \rightarrow \infty} \phi^n(x) = \infty.$$

Proof. By the assumption $|a|_p > 1$, for all $x \in \mathbb{Q}_p$ such that $|x|_p \geq 1$, we have

$$|\phi(x)|_p = \left| ax + \frac{1}{x} \right|_p = |ax|_p > |x|_p.$$

Thus the absolute values of the iterations $\phi^n(x)$ are strictly increasing. Hence

$$\lim_{n \rightarrow \infty} \phi^n(x) = \infty, \quad \text{for all } x \in \mathbb{Q}_p, |x|_p \geq 1. \quad (3.1)$$

That is to say, $\{x \in \mathbb{Q}_p : |x|_p \geq 1\}$ is included in the attracting basin of ∞ .

Now we investigate the points in the open disk $\{x \in \mathbb{Q}_p : |x|_p < 1\}$. We partition this disk into two:

$$A_1 := \left\{ x \in \mathbb{Q}_p : |x|_p < 1, |ax|_p \neq \frac{1}{|x|_p} \right\}, \quad A_2 := \left\{ x \in \mathbb{Q}_p : |x|_p < 1, |ax|_p = \frac{1}{|x|_p} \right\}.$$

If $x \in A_1$, then

$$|\phi(x)|_p = \max \left\{ |ax|_p, \frac{1}{|x|_p} \right\} > 1.$$

Thus by (3.1), $\phi(x)$ falls into the attracting basin of ∞ , and $\lim_{n \rightarrow \infty} \phi^n(x) = \infty$.

If $|ax|_p = \frac{1}{|x|_p}$, then $|a|_p = \frac{1}{|x|_p^2}$, which means that $v_p(a)$ is an even number. Since $|a|_p > 1$, the condition $\sqrt{1-a} \notin \mathbb{Q}_p$ is equivalent to that $\sqrt{-a}$ does not exist in \mathbb{Q}_p . Hence the equation $\phi(x) = 0$ has no solution in \mathbb{Q}_p . Since $p \geq 3$, this is also equivalent to that the first digits of ax and $\frac{1}{x}$ in their p -adic expansions can not be canceled. Thus $|\phi(x)|_p = |ax|_p$ or $|1/x|_p$ and hence strictly larger than 1. Therefore, by (3.1), $\lim_{n \rightarrow \infty} \phi^n(x) = \infty$. \square

Lemma 2. Suppose $|a|_p > 1$. If $\sqrt{1-a} \in \mathbb{Q}_p$, then ϕ has two repelling fixed points

$$x_{1,2} = \pm \frac{1}{\sqrt{1-a}}.$$

Proof. It is easy to check that $x_{1,2} = \pm \frac{1}{\sqrt{1-a}}$ are the two fixed points of ϕ . Note that

$$\phi'(x_1) = \phi'(x_2) = 2a - 1.$$

Since $|a|_p > 1$, we have

$$|\phi'(x_1)|_p = |\phi'(x_2)|_p = |2a - 1|_p > 1.$$

\square

Lemma 3. Suppose $|a|_p > 1$. If $\sqrt{1-a} \in \mathbb{Q}_p$, then

$$\phi(D(x_1, p^{\frac{v_p(a)}{2}-1})) = \phi(D(x_2, p^{\frac{v_p(a)}{2}-1})) = D(0, p^{-\frac{v_p(a)}{2}-1}).$$

Proof. Since $|a|_p > 1$ and $\sqrt{1-a} \in \mathbb{Q}_p$, the valuation $v_p(a)$ is a negative even number. Note that

$$|\phi(x_1)|_p = |\phi(x_2)|_p = |x_1|_p = |x_2|_p = p^{\frac{v_p(a)}{2}}.$$

We need only to show that for all x, y in the same disk $D(x_1, p^{\frac{v_p(a)}{2}-1})$ or $D(x_2, p^{\frac{v_p(a)}{2}-1})$,

$$|\phi(x) - \phi(y)|_p = p^{-v_p(a)} |x - y|_p.$$

Without loss of generality, we assume that $x, y \in D(x_1, p^{\frac{v_p(a)}{2}-1})$. By the definition of the spherical metric on $\mathbb{P}^1(\mathbb{Q}_p)$,

$$D(x_1, p^{\frac{v_p(a)}{2}-1}) = \left\{ \frac{1}{x} : |x - \sqrt{1-a}|_p \leq p^{-\frac{v_p(a)}{2}-1} \right\}.$$

Hence, there are $x', y' \in D(0, p^{-\frac{v_p(a)}{2}-1})$ such that

$$x = \frac{1}{\sqrt{1-a} + x'} \quad \text{and} \quad y = \frac{1}{\sqrt{1-a} + y'}.$$

So we have

$$\left| a - \frac{1}{xy} \right|_p = |2a - 1 - (x' + y')\sqrt{1-a} - x'y'|_p.$$

Observing that $|a|_p \geq 1$ and $|(x' + y')\sqrt{1-a}|_p \leq |a|_p/p$, we have

$$\left| a - \frac{1}{xy} \right|_p = |a|_p.$$

Hence,

$$|\phi(x) - \phi(y)|_p = \left| \left(a - \frac{1}{xy} \right) (x - y) \right|_p = |a|_p |x - y|_p = p^{-v_p(a)} |x - y|_p.$$

□

Lemma 4. Suppose $|a|_p > 1$. If $\sqrt{1-a} \in \mathbb{Q}_p$, then for all

$$x \notin D(x_1, p^{\frac{v_p(a)}{2}-1}) \cup D(x_2, p^{\frac{v_p(a)}{2}-1}),$$

we have

$$\lim_{n \rightarrow \infty} \phi^n(x) = \infty.$$

Proof. Note that for all $x \notin D(0, 1/p)$, $|1/x|_p \leq 1$. Thus

$$|\phi(x)|_p = |a|_p |x|_p > 1.$$

Hence

$$|\phi^n(x)|_p = |a|_p^n |x|_p,$$

which implies

$$\forall x \notin D(0, 1), \quad \lim_{n \rightarrow \infty} \phi^n(x) = \infty.$$

To finish the proof, we will show that for all

$$0 \neq x \in D(0, 1) \setminus (D(x_1, p^{\frac{v_p(a)}{2}-1}) \cup D(x_2, p^{\frac{v_p(a)}{2}-1})),$$

we have $|\phi(x)|_p > 1$.

We distinguish three cases.

Case (1) $|x|_p < 1/\sqrt{|a|_p}$. Since $|1/x|_p > |ax|_p$, we have

$$|\phi(x)|_p = |1/x|_p > 1.$$

Case (2) $|x|_p = 1/\sqrt{|a|_p}$. Observe that $|ax|_p = |1/x|_p = \sqrt{|a|_p}$. The assumption $x \notin D(x_1, p^{\frac{v_p(a)}{2}-1}) \cup D(x_2, p^{\frac{v_p(a)}{2}-1})$ implies that the sum of the first digits of the p -adic expansion of ax and $1/x$ is not 0 modulo p , which leads to

$$|\phi(x)|_p = |ax + 1/x|_p = |ax|_p = \sqrt{|a|_p} > 1.$$

Case (3) $1/\sqrt{|a|_p} < |x|_p < 1$. Since $|ax|_p > |1/x|_p > 1$, we have

$$|\phi(x)|_p = |ax|_p > 1.$$

□

Let (Σ_2, σ) be the full shift of two symbols.

Proposition 2. Suppose $|a|_p > 1$ and $\sqrt{1-a} \in \mathbb{Q}_p$. Then there exists an invariant set \mathcal{J} such that (\mathcal{J}, ϕ) is topologically conjugate to (Σ_2, σ) , and

$$\lim_{n \rightarrow \infty} \phi^n(x) = \infty, \quad \forall x \in \mathbb{Q}_p \setminus \mathcal{J}.$$

Proof. By the proof of Lemma 3, we obtain that both of the restricted maps

$$\phi : D(x_i, p^{\frac{v_p(a)}{2}-1}) \rightarrow D(0, p^{-\frac{v_p(a)}{2}-1}), \quad i = 1, 2$$

are expanding and bijective. Note that $D(x_i, p^{\frac{v_p(a)}{2}-1}) \subset D(0, p^{-\frac{v_p(a)}{2}-1})$ for $i = 1, 2$. Let

$$\Omega = D(x_1, p^{\frac{v_p(a)}{2}-1}) \cup D(x_2, p^{\frac{v_p(a)}{2}-1})$$

and

$$\mathcal{J} = \bigcap_{i=0}^{\infty} \phi^{-i}(\Omega).$$

By Theorem 1.1 of [15], \mathcal{J} is ϕ -invariant and (\mathcal{J}, ϕ) is topologically conjugate to (Σ_2, σ) . Note that all $x \in \Omega \setminus \mathcal{J}$ will eventually fall into $\mathbb{Q}_p \setminus \Omega$ by iteration of ϕ . Thus by Lemma 4, we immediately get

$$\lim_{n \rightarrow \infty} \phi^n(x) = \infty, \quad \forall x \in \mathbb{Q}_p \setminus \mathcal{J}.$$

□

Proof of Theorem 3. It follows directly from Propositions 1 and 2. □

3.2. Case $|a|_p < 1$. We distinguish two sub cases: $\sqrt{-a} \notin \mathbb{Q}_p$ and $\sqrt{-a} \in \mathbb{Q}_p$.

3.2.1. $\sqrt{-a} \notin \mathbb{Q}_p$.

Lemma 5. If $\sqrt{-a} \notin \mathbb{Q}_p$, then for all $-\lfloor v_p(a)/2 \rfloor \leq i \leq \lfloor v_p(a)/2 \rfloor$,

$$\phi(S(0, p^i)) \subset S(0, p^{-i})$$

and ϕ^2 is 1-Lipschitz continuous on $S(0, p^i) \cup S(0, p^{-i})$.

Proof. If $x \in S(0, p^i)$ for some $-\lfloor v_p(a)/2 \rfloor \leq i \leq 0$, then by the assumption $|a|_p < 1$, we have

$$|\phi(x)|_p = |ax + 1/x|_p = p^{-i}.$$

Now let $x \in S(0, p^i)$ for some $0 \leq i \leq \lfloor v_p(a)/2 \rfloor$. When $i < \lfloor v_p(a)/2 \rfloor$, we have

$$|\phi(x)|_p = |ax + 1/x|_p = p^{-i}.$$

When $v_p(a)/2$ is even and $i = v_p(a)/2$, the condition $\sqrt{-a} \notin \mathbb{Q}_p$ implies that

$$|\phi(x)|_p = |ax + 1/x|_p = p^{-i}.$$

Hence, the first assertion of the lemma holds.

Let us show that ϕ^2 is 1-Lipschitz continuous on $S(0, p^i) \cup S(0, p^{-i})$ for $-\lfloor v_p(a)/2 \rfloor \leq i \leq \lfloor v_p(a)/2 \rfloor$. Let $x, y \in S(0, p^i) \cap S(0, p^{-i})$. If $x \in S(0, p^i)$ and $y \in S(0, p^{-i})$, then $|xy|_p = 1$ and thus

$$|\phi(x) - \phi(y)|_p = \left| a - \frac{1}{xy} \right|_p |x - y|_p = |x - y|_p.$$

Hence, it suffices to show that for each $-\lfloor v_p(a)/2 \rfloor \leq i \leq \lfloor v_p(a)/2 \rfloor$,

$$\forall x, y \in S(0, p^i), \quad |\phi^2(x) - \phi^2(y)|_p \leq |x - y|_p.$$

By observing that $|a|_p \leq 1/|xy|_p$ and $|a|_p \leq 1/|\phi(x)\phi(y)|_p$, we have

$$\begin{aligned} |\phi^2(x) - \phi^2(y)|_p &= \left| a - \frac{1}{\phi(x)\phi(y)} \right|_p |\phi(x) - \phi(y)|_p \\ &= \left| a - \frac{1}{\phi(x)\phi(y)} \right|_p \left| a - \frac{1}{xy} \right|_p |x - y|_p \\ &\leq \left| \frac{1}{\phi(x)\phi(y)} \right|_p \left| \frac{1}{xy} \right|_p |x - y|_p = |x - y|_p. \end{aligned}$$

□

The field \mathbb{C}_p of p -adic complex numbers is the metric completion of the algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p . We denote by $B(a, r) := \{x \in \mathbb{C}_p, |x|_p \leq r\}$ the closed ball in \mathbb{C}_p centered at a with radius $r > 0$.

Lemma 6. Assume $\sqrt{-a} \notin \mathbb{Q}_p$. Let $x_0 \in \mathbb{Q}_p$ with $p^{-\lfloor v_p(a)/2 \rfloor} \leq |x_0|_p \leq p^{\lfloor v_p(a)/2 \rfloor}$. Then $\phi^2(B(x_0, |x_0|_p/p)) \subset B(\phi^2(x_0), |x_0|_p/p)$.

Proof. For any $x \in B(x_0, |x_0|_p/p)$, the condition $\sqrt{-a} \notin \mathbb{Q}_p$ implies

$$|x\phi(x)|_p = 1.$$

Hence

$$\begin{aligned} |\phi^2(x) - \phi^2(x_0)|_p &= \left| a - \frac{1}{\phi(x)\phi(x_0)} \right|_p |\phi(x) - \phi(x_0)|_p \\ &= \left| a - \frac{1}{\phi(x)\phi(x_0)} \right|_p \left| a - \frac{1}{xx_0} \right|_p |x - x_0|_p \\ &\leq \left| \frac{1}{\phi(x)\phi(x_0)} \right|_p \left| \frac{1}{xx_0} \right|_p |x - x_0|_p = |x - x_0|_p. \end{aligned}$$

□

Lemma 7. Assume $\sqrt{-a} \notin \mathbb{Q}_p$. Let $x_0 \in \mathbb{Q}_p$ with $p^{-\lfloor v_p(a)/2 \rfloor} \leq |x_0|_p \leq p^{\lfloor v_p(a)/2 \rfloor}$. Then the Taylor expansion of the map

$$\phi^2 : D(x_0, |x_0|_p/p) \rightarrow D(\phi^2(x_0), |x_0|_p/p)$$

can be written as

$$\phi^2(x + x_0) = \phi^2(x_0) + \sum_{i=1}^{\infty} \alpha_i (x + x_0)^i,$$

with $\alpha_i \in \mathbb{Q}_p$ such that

$$|\alpha_i|_p \leq (p/|x_0|_p)^{i-1}.$$

Proof. The assumption $\sqrt{-a} \notin \mathbb{Q}_p$ implies that there is no ϕ -preimage of 0 in \mathbb{Q}_p . However, there are two preimages $\pm \frac{1}{\sqrt{-a}}$ of 0 in the quadratic extension $\mathbb{Q}_p(\sqrt{-a})$. For each $x_0 \in \mathbb{Q}_p$ with $p^{-\lfloor v_p(a)/2 \rfloor} \leq |x_0|_p \leq p^{\lfloor v_p(a)/2 \rfloor}$, the condition $\sqrt{-a} \notin \mathbb{Q}_p$ implies that

$$\left| x_0 - \frac{1}{\sqrt{-a}} \right|_p = \left| x_0 + \frac{1}{\sqrt{-a}} \right|_p > \frac{|x_0|_p}{p}.$$

By Lemma 5, we also have

$$\left| \phi(x_0) - \frac{1}{\sqrt{-a}} \right|_p = \left| \phi(x_0) + \frac{1}{\sqrt{-a}} \right|_p > \frac{|\phi(x_0)|_p}{p}.$$

Hence, the ball

$$B(x_0, |x_0|_p/p) := \left\{ x \in \mathbb{C}_p, |x|_p \leq \frac{|x_0|_p}{p} \right\}$$

is disjoint from the set $\{\pm \frac{1}{\sqrt{-a}}, 0, \infty\}$ of polars of ϕ^2 . This implies that the Taylor expansion of ϕ^2 at x_0

$$\phi^2(x + x_0) = \phi^2(x_0) + \sum_{i=1}^{\infty} \alpha_i (x + x_0)^i,$$

is convergent on the ball $B(x_0, |x_0|_p/p)$. By Lemma 6, we have $\phi^2(B(x_0, |x_0|_p/p)) \subset B(\phi^2(x_0), |x_0|_p/p)$. The Newton polygon [26, p. 249] gives

$$|\alpha_i|_p \left(\frac{|x_0|_p}{p} \right)^i \leq \frac{|x_0|_p}{p}.$$

That is $|\alpha_i|_p \leq (p/|x_0|_p)^{i-1}$. \square

Lemma 8. *If $\sqrt{-a} \notin \mathbb{Q}_p$, then for each $x \in \mathbb{Q}_p \setminus \{0, \infty\}$, there exists a positive integer N such that*

$$\phi^n(x) \in S(0, p^i) \cup S(0, p^{-i}), \quad \forall n \geq N$$

for some $0 \leq i \leq \lfloor v_p(a)/2 \rfloor$.

Proof. Note that $|\phi(x)|_p = |1/x|_p$, if $|x|_p < p^{-\lfloor v_p(a)/2 \rfloor}$. Thus we have

$$\phi(D(0, p^{-\lfloor v_p(a)/2 \rfloor - 1})) = \mathbb{P}^1(\mathbb{Q}_p) \setminus D(0, p^{\lfloor v_p(a)/2 \rfloor}).$$

It suffices to show that the statement holds for $x \in \mathbb{P}^1(\mathbb{Q}_p) \setminus D(0, p^{\lfloor v_p(a)/2 \rfloor})$. In fact, if $x \in \mathbb{P}^1(\mathbb{Q}_p) \setminus D(0, p^{\lfloor v_p(a)/2 \rfloor + 1})$, one can check that

$$|\phi(x)|_p = |a|_p |x|_p.$$

So there exists an integer N such that $p^{-\lfloor v_p(a)/2 \rfloor} \leq |\phi^N(x)|_p \leq p^{\lfloor v_p(a)/2 \rfloor}$. If $x \in S(0, p^{\lfloor v_p(a)/2 \rfloor})$, the conclusion is followed by Lemma 5. \square

Proposition 3. *Assume that $\sqrt{-a} \notin \mathbb{Q}_p$. Then $\mathbb{Q}_p \setminus \{0\}$ is ϕ -invariant and the sub dynamical system $(\mathbb{Q}_p \setminus \{0\}, \phi)$ can be decomposed into*

$$\mathbb{P}^1(\mathbb{Q}_p) = \mathcal{P} \bigsqcup \mathcal{M} \bigsqcup \mathcal{B}$$

where \mathcal{P} is the finite set consisting of all periodic points of ϕ , $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each \mathcal{M}_i is a finite union of balls and each subsystem $\phi : \mathcal{M}_i \rightarrow \mathcal{M}_i$ is minimal, and each point in \mathcal{B} lies in the attracting basin of a periodic orbit or of a minimal subsystem.

Proof. By Lemmas 5 and 8, it suffices to show that the sub dynamical system

$$(S(0, p^i), \phi^2)$$

has a minimal decomposition as stated in the proposition for all $-\lfloor v_p(a)/2 \rfloor \leq i \leq \lfloor v_p(a)/2 \rfloor$. Note that ϕ^2 is equicontinuous on $S(0, p^i)$ and

$$S(0, p^i) = \bigcup_{j=1}^{p-1} D(jp^{-i}, p^{i-1}).$$

Thus for each disk $D(jp^{-i}, p^{i-1})$ in $S(0, p^i)$, its image $\phi^2(D(jp^{-i}, p^{i-1}))$ is still a disk in $S(0, p^i)$. Hence ϕ^2 induces a map $\overline{\phi^2}$ from the set $\{D(jp^{-i}, p^{i-1}) : j = \{1, 2, \dots, p-1\}\}$ of disks to itself:

$$\overline{\phi^2}(D(jp^{-i}, p^{i-1})) := \phi^2(D(jp^{-i}, p^{i-1})).$$

Assume that $(D(x_1p^{-i}, p^{i-1}), D(x_2p^{-i}, p^{i-1}), \dots, D(x_kp^{-i}, p^{i-1}))$ is a k -cycle of $\overline{\phi^2}$, i.e.

$$\begin{aligned} \overline{\phi^2}(D(x_1p^{-i}, p^{i-1})) &= D(x_2p^{-i}, p^{i-1}) \\ \overline{\phi^2}(D(x_2p^{-i}, p^{i-1})) &= D(x_3p^{-i}, p^{i-1}), \\ &\dots \\ \overline{\phi^2}(D(x_kp^{-i}, p^{i-1})) &= D(x_1p^{-i}, p^{i-1}). \end{aligned}$$

Then ϕ^{2k} is a transformation on $D(x_1p^{-i}, p^{i-1})$. We thus study the dynamical system $(D(x_1p^{-i}, p^{i-1}), \phi^{2k})$. By Lemma 8, the Taylor expansion of the map

$$\phi^{2k} : D(x_1p^{-i}, p^{i-1}) \rightarrow D(x_1p^{-i}, p^{i-1})$$

can be written as

$$\phi^{2k}(x + x_1p^{-i}) = \phi^{2k}(x_1p^{-i}) + \sum_{j=1}^{\infty} \alpha_j (x + x_1p^{-i})^j,$$

with $\alpha_j \in \mathbb{Q}_p$ such that

$$|\alpha_j|_p \leq 1/p^{(i-1)(j-1)}. \quad (3.2)$$

Thus one can check that the dynamical system $(D(x_1p^{-i}, p^{i-1}), \phi^{2k})$ is conjugate to a dynamical system on \mathbb{Z}_p , denoted by (\mathbb{Z}_p, ψ) , through the conjugacy

$$f : D(x_1p^{-i}, p^{i-1}) \rightarrow D(0, 1), \text{ with } f(x) = p^{i-1}(x - x_1p^{-i}).$$

In fact, the inequality (3.2) implies that ψ is a convergent series with integer coefficients. By Theorem 1.1 of [16], the system (\mathbb{Z}_p, ψ) has a minimal decomposition. Hence, the conjugated dynamical system $(D(x_1p^{-i}, p^{i-1}), \phi^{2k})$ has a corresponding minimal decomposition which implies that

$$(S(0, p^i), \phi^2)$$

has a minimal decomposition as stated in the proposition for all $-\lfloor v_p(a)/2 \rfloor \leq i \leq \lfloor v_p(a)/2 \rfloor$. □

3.2.2. Case $\sqrt{-a} \in \mathbb{Q}_p$.

Lemma 9. *If $|a| < 1$ and $\sqrt{-a} \in \mathbb{Q}_p$, then*

$$\phi \left(D \left(\pm \frac{1}{\sqrt{-a}}, \frac{1}{p\sqrt{|a|_p}} \right) \right) = D \left(0, \frac{\sqrt{|a|_p}}{p} \right),$$

and for all $x, y \in D(\pm \frac{1}{\sqrt{-a}}, \frac{1}{p\sqrt{|a|_p}})$,

$$|\phi(x) - \phi(y)|_p = |a|_p |x - y|_p.$$

.

Proof. Note that $\phi(\pm 1/\sqrt{-a}) = 0$. It suffices to show that

$$\forall x, y \in D(\pm \frac{1}{\sqrt{-a}}, \frac{1}{p\sqrt{|a|_p}}), \quad |\phi(x) - \phi(y)|_p = |a|_p |x - y|_p.$$

Without loss of generality, we assume that $x, y \in D(\frac{1}{\sqrt{-a}}, \frac{1}{p\sqrt{|a|_p}})$. By the same arguments in the proof of Lemma 3, there exist $x', y' \in D(0, \frac{\sqrt{|a|_p}}{p})$ such that

$$x = \frac{1}{\sqrt{-a} + x'}, \quad y = \frac{1}{\sqrt{-a} + y'},$$

and

$$\left| a - \frac{1}{xy} \right|_p = |2a - (x' + y')\sqrt{-a} - x'y'|_p.$$

Since $x', y' \in D(0, \frac{\sqrt{|a|_p}}{p})$, we immediately get

$$\left| a - \frac{1}{xy} \right|_p = |a|_p.$$

So

$$|\phi(x) - \phi(y)|_p = \left| a - \frac{1}{xy} \right|_p |x - y|_p = |a|_p |x - y|_p.$$

□

Proposition 4. *If $|a|_p < 1$ and $\sqrt{-a} \in \mathbb{Q}_p$, then there exists an ϕ -invariant subset \mathcal{J} such that (\mathcal{J}, ϕ) is topologically conjugate to a subshift of finite type (Σ_A, σ) , where the transition matrix is*

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

The topological entropy of the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is larger than $\log 1.69562\dots$

Proof. Let $D_1 = D(1/\sqrt{-a}, 1/p)$, $D_2 = D(-1/\sqrt{-a}, 1/p)$, $D_3 = D(0, |a|_p/p)$ and $D_4 = \mathbb{P}^1(\mathbb{Q}_p) \setminus D(0, |a|_p)$. By Lemma 9, the restricted maps $\phi : D_1 \rightarrow D_3$ and $\phi : D_2 \rightarrow D_3$ are both bijective. One can also check directly that $\phi : D_3 \rightarrow D_4$ and $\phi : D_4 \rightarrow \phi(D_4) = \mathbb{P}^1(\mathbb{Q}_p) \setminus D(0, 1)$ are bijective.

Set $\Omega = \bigcup_{i=1}^4 D_i$ and consider the restricted map $\phi : \Omega \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$. Let $f : x \mapsto 1/(x - 1)$ and $\psi := f \circ \phi \circ f^{-1}$. We study the map $\psi : f(\Omega) \rightarrow \mathbb{P}^1(\mathbb{Q}_p)$. Observe that $f(\Omega) \subset \mathbb{Z}_p$. One can check that ψ satisfies the conditions in [15]. Thus by the main result of [15],

$$\mathcal{J}' = \bigcap_{i=0} \psi^{-i}(\Omega)$$

is an invariant set of ψ and (\mathcal{J}', ψ) is topologically conjugate to the subshift of finite type (Σ_A, σ) with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Since $\psi = f \circ \phi \circ f^{-1}$, we deduce that (\mathcal{J}, ϕ) is topologically conjugate to (Σ_A, σ) .

The topological entropy of (Σ_A, σ) is $\log(1.69562\dots)$ where $1.69562\dots$ is the maximal eigenvalue of the matrix A . Since (Σ_A, σ) is topologically conjugate to a subsystem of $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$, we confirm the last assertion of the proposition. \square

We remark that even though a chaotic subsystem is well described, the detailed dynamical structure of ϕ on the whole space for the case $|a|_p < 1$ and $\sqrt{-a} \in \mathbb{Q}_p$ is far from clear. There may exist more complicated sub dynamical systems.

Proof of Theorem 4. It follows directly from Propositions 3 and 4. \square

REFERENCES

- [1] V. S. Anashin, Uniformly distributed sequences of p -adic integers, *Mat. Zametki*, 55(2):3–46, 188, 1994.
- [2] V. S. Anashin and A. Khrennikov, *Applied Algebraic Dynamics*, de Gruyter Expositions in Mathematics. 49. Walter de Gruyter & Co., Berlin, 2009.
- [3] V. S. Anashin, A. Khrennikov, and E. I. Yurova, Characterization of ergodic p -adic dynamical systems in terms of the van der Put basis, *Dokl. Akad. Nauk*, 438(2):151–153, 2011.
- [4] S. Albeverio, U. A. Rozikov and I. A. Sattarov, p -adic $(2, 1)$ -rational dynamical systems, *J. Math. Anal. Appl.*, 398 (2): 553–566, 2013.
- [5] M. Baker and R. Rumely, *Potential theory and dynamics on the Berkovich projective line*, volume 159 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2010.
- [6] R. L. Benedetto, Hyperbolic maps in p -adic dynamics, *Ergodic Theory Dynam. Systems*, 21 (1):1–11, 2001.
- [7] Z. Coelho and W. Parry, Ergodicity of p -adic multiplications and the distribution of Fibonacci numbers, In *Topology, ergodic theory, real algebraic geometry*, volume 202 of *Amer. Math. Soc. Transl. Ser. 2*, pages 51–70. Amer. Math. Soc., Providence, RI, 2001.
- [8] B. Dragovich, A. Khrennikov, and D. Mihajlović, Linear fractional p -adic and adelic dynamical systems, *Rep. Math. Phys.*, 60 (1): 55–68, 2007.
- [9] V. Dremov, G. Shabat and P. Vytova, On the chaotic properties of quadratic maps over non-Archimedean fields, *p-adic mathematical physics*, 43–54, AIP Conf. Proc., 826, Amer. Inst. Phys., Melville, NY, 2006.
- [10] F. Durand and F. Paccaut, Minimal polynomial dynamics on the set of 3-adic integers, *Bull. Lond. Math. Soc.*, 41(2):302–314, 2009.
- [11] A. H. Fan, S. L. Fan, L. M. Liao, and Y. F. Wang, On minimal decomposition of p -adic homographic dynamical systems, *Adv. Math.*, 257:92–135, 2014.
- [12] A. H. Fan, S. L. Fan, L. M. Liao, and Y. F. Wang, Minimality of p -adic rational maps with good reduction, *preprint*, arXiv:1511.04856.
- [13] A. H. Fan, M. T. Li, J. Y. Yao, and D. Zhou, Strict ergodicity of affine p -adic dynamical systems on \mathbb{Z}_p , *Adv. Math.*, 214(2):666–700, 2007.
- [14] A. H. Fan and L. M. Liao, On minimal decomposition of p -adic polynomial dynamical systems, *Adv. Math.*, 228:2116–2144, 2011.
- [15] A. H. Fan, L. M. Liao, Y. F. Wang, and D. Zhou, p -adic repellers in \mathbb{Q}_p are subshifts of finite type, *C. R. Acad. Sci. Paris, Ser. I*, 344:219–224, 2007.
- [16] S. L. Fan and L. M. Liao, Dynamics of convergent power series on the integral ring of a finite extension of \mathbb{Q}_p , *J. Differential Equations*, 259(4):1628–1648, 2015.
- [17] S. L. Fan and L. M. Liao, Dynamics of the square mapping on the ring of p -adic integers, *Proc. Amer. Math. Soc.*, 144(3):1183–1196, 2016.
- [18] S. L. Fan and L. M. Liao, Dynamics of Chebyshev polynomials on \mathbb{Z}_2 , *J. Number Theor.*, 169:174–182, 2016.
- [19] L. C. Hsia, Closure of periodic points over a non-Archimedean field, *J. London Math. Soc. (2)*, 62 (3):685–700, 2000.
- [20] S. Jeong, Toward the ergodicity of p -adic 1-Lipschitz functions represented by the van der Put series, *J. Number Theory*, 133(9):2874–2891, 2013.
- [21] M. Khamraev and F. M. Mukhamedov, On a class of rational p -adic dynamical systems, *J. Math. Anal. Appl.*, 315 (1):76–89, 2006.
- [22] K. Mahler, p -Adic Numbers and Their Functions, second edition, *Cambridge Tracts in Math.*, vol.76, Cambridge University Press, Cambridge, 1981.
- [23] F. M. Mukhamedov and U. A. Rozikov, On rational p -adic dynamical systems, *Methods Funct. Anal. Topology*, 10 (2):21–31, 2004.

- [24] R. Oseles and H. Zieschang, Ergodische Eigenschaften der Automorphismen p -adischer Zahlen, *Arch. Math. (Basel)*, 26:144–153, 1975.
- [25] J. Rivera-Letelier, Dynamique des fonctions rationnelles sur des corps locaux, *Astérisque*, (287):xv, 147–230, 2003. Geometric methods in dynamics. II.
- [26] J. Silverman, *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.
- [27] I. A. Sattarov, p -adic $(3, 2)$ -rational dynamical systems. *p-Adic Numbers Ultrametric Anal. Appl.* 7(1):39–55, 2015.
- [28] E. Thiran, D. Verstegen and J. Weyers, p -adic dynamics, *J. Statist. Phys.*, 54(3-4): 893–913, 1989.
- [29] C. F. Woodcock and N. P. Smart, p -adic chaos and random number generation, *Experiment Math.*, 7(4): 333–342, 1998.

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